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Modified Weisskopf–Schwinger Lagrangian

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Abstract. The one-loop effective potential in quantum electrodynamics is extended by including radiative corrections to the electron propagator. In particular, the limiting case of a very strong external $H(E)$ field is examined and a dissident view is presented with regard to recent investigations concerning the modified Coulomb potential at small distances. Our treatment is based on non-perturbative functional techniques.

1. Introduction

The goal of the following considerations is to further explore quantum mechanical corrections for electromagnetic processes generated by constant fields. From the work of Heisenberg and Euler (1936) and Weisskopf (1936) it is well known that classical electromagnetism becomes a non-linear theory due to the probability of pair creation (vacuum polarization effects). As a consequence, the classical expression for the Lagrangian $L=\frac{1}{2}(E^2-H^2)$ acquires corrections which cause interactions between electromagnetic fields and thus violate the superposition principle. The scattering of light or Delbrück scattering are familiar examples for this quantum mechanical non-linearity. Another highly non-linear process is the vacuum persistence amplitude in the presence of an external electromagnetic field. Physically it represents the effect that an arbitrary number of external photon lines can have on a single charged-particle loop, i.e., vacuum polarization to all orders in the prescribed electromagnetic field. It is this process which supplies the first non-linear correction to the Lagrangian of classical electromagnetism.

Among the various computations of this one-loop effective Lagrangian starting with Heisenberg and Euler (1936) and Weisskopf (1936), Schwinger (1951) has probably influenced most of the work that has been performed since. It is in Schwinger's paper where, among other important contributions, the proper-time method was employed to deal with external field problems. Dittrich (1976) and certainly several other people (e.g. Brown and Duff 1975) have invented their own methods to derive the relevant Green function of the problem. From here on one can then compute the mass operator (Tsai 1974a and references therein), the vacuum polarization tensor (Tsai 1974b and references therein), etc in the presence of a constant field. However, the original papers on quantum mechanical corrections to the classical Lagrangian of a constant electromagnetic field have always excluded radiative corrections. Only recently does one find contributions in published and unpublished papers on the subject of the Weisskopf–Schwinger Lagrangian which go beyond the one-loop approximation. A first step in this direction is contained in a preprint by V I Ritus (1975, *P N Lebedev Physical Institute, Moscow Preprint No.* 125). Also, in Greenman and Rohrlich (1973)—although still limited to the one-loop effective Lagrangian—one finds some speculations concerning radiative corrections and their impact on the existence of a maximal electrostatic field strength.

In a previous article (Dittrich 1976) we started a detailed investigation of vacuum polarization effects for different types of electromagnetic fields. Here we want to concentrate on a purely magnetic field right from the beginning. Thereafter we will go one step beyond present one-loop calculations and incorporate corrections with one internal photon line. Since we intend to produce an exact Lagrangian as far as the external field is concerned, we will formulate the present problem by means of non-perturbative functional techniques.

2. Vacuum persistence amplitude

Our present interest is devoted to the vacuum-to-vacuum amplitude (Fried 1972, Bialynicki-Birula and Bialynicki-Birula 1975)

$$N_{\mathrm{v}} = \exp\left(-\frac{\mathrm{i}}{2}\int \frac{\delta}{\delta J} D_+ \frac{\delta}{\delta J}\right) \exp(\mathrm{i}\, W[A+J])|_{J=0}, \tag{2.1}$$

where $A_\mu(x)$ represents the external field and $J_\mu(x)$ denotes a c-number source which is thought to be coupled to the photon field. Setting $J_\mu = 0$ on the right-hand side of (2.1) means that we are generating internal photon lines only. $W[A+J]$ is the well known QED action. Since we want to treat the external field $A_\mu(x)$ to all orders, we proceed by expansion of $W[A+J]$:

$$\begin{aligned} W[A+J] &= W[A] + \left(\frac{\delta}{\delta J^\mu} W[A+J]\right)_{J=0} J^\mu + \frac{1}{2}\left(\frac{\delta^2 W[A+J]}{\delta J^\mu \delta J^\nu}\right)_{J=0} J^\mu J^\nu + \dots \\ &= W^{(1)}[A] + \int (\mathrm{d}x) \langle j_\mu^A(x)\rangle J^\mu(x) + \tfrac{1}{2}\int\int (\mathrm{d}x)(\mathrm{d}y) J^\mu(x) \Pi_{\mu\nu}^A(x,y) J^\nu(y) + \dots, \end{aligned} \tag{2.2}$$

where we introduced (Dittrich 1976)

$$\mathrm{i}\, W^{(1)}[A] = -\mathrm{Tr}\ln(1 - e\gamma\,.\,AG_+)^{-1} \tag{2.3}$$

and furthermore

$$\langle j_\mu^A(x)\rangle = \mathrm{i}e\ \mathrm{tr}[\gamma_\mu G_+(x,x|A)], \tag{2.4}$$

$$\Pi_{\mu\nu}(x,y|A) = \mathrm{i}e^2\ \mathrm{tr}[\gamma_\mu G_+(x,y|A)\gamma_\nu G_+(y,x|A)]. \tag{2.5}$$

Accordingly we obtain

$$\begin{aligned} N_{\mathrm{v}} &= \exp(\mathrm{i}\, W^{(1)}[A]) \exp\left(-\frac{\mathrm{i}}{2}\int \frac{\delta}{\delta J} D_+ \frac{\delta}{\delta J}\right) \exp\left(\frac{\mathrm{i}}{2}\int J^\mu \Pi_{\mu\nu} J^\nu\right) \exp\left(\mathrm{i}\int \langle j_\mu^A\rangle J^\mu\right)\Big|_{J=0} \\ &= \exp(\mathrm{i}\, W^{(1)}[A]) \exp[\tfrac{1}{2}\mathrm{Tr}\ln(1 - D_+\Pi)^{-1} + \tfrac{1}{2}\mathrm{i}\langle j^A\rangle D_+(1-\Pi D_+)^{-1}\langle j^A\rangle]. \end{aligned} \tag{2.6}$$

If we expand the Tr ln term, we have

$$N_v = \exp\left(\mathrm{i}\int (\mathrm{d}x)\mathscr{L}^{(1)}(x)\right)$$
$$\times\exp\left[\mathrm{i}\int (\mathrm{d}x)\left(\frac{e^2}{2}\int \mathrm{d}y\ \mathrm{tr}[\gamma_\mu G_+(x,y|A)\gamma_\nu G_+(y,x|A)]D_+^{\mu\nu}(x-y)\right)\right]$$
$$\times\exp(\tfrac{1}{2}\mathrm{i}\langle j^A\rangle D_+\langle j^A\rangle). \tag{2.7}$$

It is the first term beyond $\mathscr{L}^{(1)}(x)$ on which we want to focus our attention. While the first-order non-linear correction $\mathscr{L}^{(1)}(x)$ is graphically represented by figure 1, the second-order correction to the Lagrange function is given by figure 2. Bold lines indicate the interaction of the virtual electron to all orders in the external field. Notice, however, that the second graph in figure 2 does not contribute, since $\langle j_\mu^A(x)\rangle$ is zero for a constant field. Our entire second-order non-linear correction is therefore correctly taken into account by the single internal loop correction.

Figure 1. First-order non-linear correction to the Lagrangian.

Figure 2. Second-order correction to the Lagrangian.

3. Radiative corrections, $\mathscr{L}^{(2)}(x)$

In this section we want to compute

$$\mathscr{L}^{(2)}(x) = \tfrac{1}{2}e^2\int (\mathrm{d}x')\ \mathrm{tr}[\gamma_\mu G_+(x,x'|A)\gamma_\nu G_+(x',x|A)]D_+^{\mu\nu}(x-x'). \tag{3.1}$$

To accomplish this goal we need the electron's Green function in the presence of an external electromagnetic field $A_\mu(x)$. In what follows we concentrate on a constant magnetic (electric) field. For this case the Green function allows for a closed-form expression which was re-examined in Dittrich (1976) using straightforward Green functions techniques in momentum space. The result is stated in

$$G_+(x,x'|A) = \phi(x,x')\left[m - \gamma^\mu\left(\frac{1}{\mathrm{i}}\partial_\mu^x + \frac{e}{2}(x-x')^\nu F_{\mu\nu}\right)\right]\Delta_+(x,x'|A'), \tag{3.2}$$

where $A'_\mu = -\tfrac{1}{2}(x-x')^\nu F_{\mu\nu}$,

$$\Delta_+(x,x'|A') = \frac{1}{(4\pi)^2}\int_0^\infty \frac{\mathrm{d}s}{s^2}\exp[-\mathrm{i}s(m^2 - \tfrac{1}{2}e\sigma F)]\,\mathrm{e}^{-L(s)}$$
$$\times\exp\{\tfrac{1}{4}\mathrm{i}(x-x')[eF\coth(eFs)](x-x')\} \tag{3.3}$$

and

$$\phi(x, x') = \exp\left(ie \int_{x'}^{x} d\xi^\mu A_\mu(\xi)\right). \tag{3.4}$$

The loop factor $L(s)$ is given by

$$L(s) = \tfrac{1}{2} \operatorname{tr} \ln[(eFs)^{-1} \sinh(eFs)].$$

If there is only a magnetic field present, which we assume to be in the z direction, $F_{12} = -F_{21} = H$, we obtain

$$e^{-L(s)} = eHs/\sin(eHs). \tag{3.5}$$

This yields the Green function in configuration space

$$G_+(x, x'|A) = \phi(x, x') \frac{1}{(4\pi)^2} \int_0^\infty \frac{ds}{s^2} [m - \tfrac{1}{2}\gamma^\mu (f(s) + eF)_{\mu\nu} (x - x')^\nu]$$
$$\times \exp[-im^2 s - L(s) + \tfrac{1}{4} i (x - x') f(s) (x - x')] \, e^{\frac{1}{2} ie\sigma Fs} \tag{3.6}$$

where we have introduced $f(s) = eF \coth(eFs)$. Substituting the propagation function for the free photon:

$$D_+(x - x') = (4\pi)^{-2} \int_0^\infty \exp[\tfrac{1}{4} i (x - x')^2 / t] \, dt/t^2 \tag{3.7}$$

(we have chosen the Feynman gauge $D_{+\mu\nu} = g_{\mu\nu} D_+$), and noting that $\phi(x, x')\phi(x', x) = 1$, $\mathscr{L}^{(2)}$ can be written as

$$\mathscr{L}^{(2)} = \frac{e^2}{2(4\pi)^6} \int_0^\infty \frac{ds_1}{s_1^2} \int_0^\infty \frac{ds_2}{s_2^2} \int_0^\infty \frac{dt}{t^2} \exp[-im^2(s_1 + s_2) - L(s_1) - L(s_2)]$$
$$\times \int (dz) \exp\{\tfrac{1}{4} iz (f(s_1) + f(s_2) + t^{-1}) z\} \operatorname{tr}(\ldots), \tag{3.8}$$

with

$$\operatorname{tr}(\ldots) = \operatorname{tr}\{\gamma^\mu [m - \tfrac{1}{2}\gamma (f(s_1) + eF) z] \, e^{\frac{1}{2} ie\sigma Fs_1} \gamma_\mu [m + \tfrac{1}{2}\gamma (f(s_2) + eF) z] \, e^{\frac{1}{2} ie\sigma Fs_2}\}. \tag{3.9}$$

In order to continue our calculation we need the following traces in spinor space:

$$m^2 \operatorname{tr}(\gamma^\mu \, e^{\frac{1}{2} ie\sigma Fs_1} \gamma_\mu \, e^{\frac{1}{2} ie\sigma Fs_2}) = -16\, m^2 \cos(eHs_1) \cos(eHs_2) \tag{3.10}$$

$$\operatorname{tr}(\gamma^\mu \gamma_\rho \, e^{\frac{1}{2} ie\sigma Fs_1} \gamma_\mu \gamma_\lambda \, e^{\frac{1}{2} ie\sigma Fs_2}) = -8 \cos[eH(s_2 - s_1)] g_{\lambda\rho} - 8 \sin[eH(s_2 - s_1)] (eF/eH)_{\lambda\rho}. \tag{3.11}$$

Introducing these results in equation (3.9) we arrive at

$$\operatorname{tr}(\ldots) = -2\{8m^2 \cos x_1 \cos x_2 - (f_2^T - eF)_{\tau\lambda} [g_{\lambda\rho} C + (eF/eH)_{\lambda\rho} S] (f_1 + eF)_{\rho\sigma} z^\sigma z^\tau\}, \tag{3.12}$$

where

$$x_1 = eHs_1, \qquad x_2 = eHs_2$$
$$C = \cos(x_2 - x_1), \qquad S = \sin(x_2 - x_1).$$

Equation (3.8) calls for the integral

$$\int (dz) \exp\{\tfrac{1}{4} iz (f_1 + f_2 + t^{-1}) z\} \operatorname{tr}(\ldots), \tag{3.13}$$

whose value we can read off from

$$\int (\mathrm{d}z)\, \exp(\tfrac{1}{4}\mathrm{i}zXz) = \mathrm{i}(4\pi)^2/(\det X)^{1/2},$$

and

$$\int (\mathrm{d}z)\, \exp(\tfrac{1}{4}\mathrm{i}zXz) z_\sigma z_\tau = \mathrm{i}\frac{(4\pi)^2}{(\det X)^{1/2}} 2\mathrm{i}(X^{-1})_{\sigma\tau}.$$

According to these formulae, we find

$$\int (\mathrm{d}z)\, \exp\{\tfrac{1}{4}\mathrm{i}z(f(s_1)+f(s_2)+t^{-1})z\}\,\mathrm{tr}(\ldots)$$
$$= -\mathrm{i}\frac{4(4\pi)^2}{(\det X)^{1/2}} [\![4m^2 \cos x_1 \cos x_2 - \mathrm{i}\,\mathrm{tr}'\{(f^{\mathrm{T}}(s_2)-eF)[C+(eF/eH)S] \times (f(s_1)+eF)X^{-1}\}]\!]$$

where tr′ means trace in Minkowski μ, ν space.

$(\det X)^{1/2}$ is given by

$$(\det X)^{1/2} = [\det(f(s_1)+f(s_2)+t^{-1})]^{1/2} = \Big(\prod_{\substack{\text{eigen-}\\\text{values}}} (f(s_1)+\ldots)\Big)^{1/2},$$

which, for a purely magnetic field, yields

$$(\det X)^{1/2} = (\alpha + t^{-1})(\beta + t^{-1})$$

with

$$\alpha = eH[\cot(eHs_1)+\cot(eHs_2)] \qquad \text{and} \qquad \beta = \frac{1}{s_1}+\frac{1}{s_2}.$$

For the trace tr′ we obtain

$$\mathrm{tr}'(\ldots) = \frac{1}{\alpha+t^{-1}} g(s_1, s_2) + \frac{1}{\beta+t^{-1}} h(s_1, s_2)$$

where

$$g(s_1, s_2) = \frac{2(eH)^2}{\sin(eHs_1)\sin(eHs_2)} \qquad \text{and} \qquad h(s_1, s_2) = \frac{2}{s_1 s_2}\cos[eH(s_2-s_1)].$$

At last we find for $\mathscr{L}^{(2)}$

$$\mathscr{L}^{(2)} = -\mathrm{i}\frac{\alpha}{32\pi^3}\int_0^\infty \frac{\mathrm{d}s_1}{s_1}\frac{\mathrm{d}s_2}{s_2}\frac{\mathrm{e}^{-\mathrm{i}m^2(s_1+s_2)}}{\sin(eHs_1)\sin(eHs_2)}(eH)^2\bigg[\int_0^\infty \frac{\mathrm{d}t}{t^2}\frac{4m^2\cos(eHs_1)\cos(eHs_2)}{(\alpha+t^{-1})(\beta+t^{-1})}$$
$$-\mathrm{i}\int_0^\infty \frac{\mathrm{d}t}{t^2}\bigg(g(s_1,s_2)\frac{1}{(\alpha+t^{-1})^2}\frac{1}{\beta+t^{-1}} + h(s_1,s_2)\frac{1}{\alpha+t^{-1}}\frac{1}{(\beta+t^{-1})^2}\bigg)\bigg].$$

The t integrals are elementary and lead to

$$I_1=\int_0^\infty \frac{dt}{t^2}\frac{1}{\alpha+t^{-1}}\frac{1}{\beta+t^{-1}}=\frac{1}{\beta-\alpha}\ln\left(\frac{\beta}{\alpha}\right),$$

$$I_2=\int_0^\infty \frac{dt}{t^2}\left(\frac{g}{(\alpha+t^{-1})^2(\beta+t^{-1})}+\frac{h}{(\alpha+t^{-1})(\beta+t^{-1})^2}\right)=-\frac{g-h}{(\beta-\alpha)^2}\ln\left(\frac{\beta}{\alpha}\right)-\frac{\alpha h-\beta g}{\alpha\beta(\beta-\alpha)}.$$

In terms of these integrals we get

$$\mathscr{L}^{(2)}=-\mathrm{i}\frac{\alpha}{32\pi^3}(eH)^2\int_{\to 0}^\infty \frac{ds_1}{s_1}\int_{\to 0}^\infty \frac{ds_2}{s_2}\frac{\exp[-\mathrm{i}m^2(s_1+s_2)]}{\sin(eHs_1)\sin(eHs_2)}$$
$$\times[4m^2\cos(eHs_1)\cos(eHs_2)I_1-\mathrm{i}I_2]+\mathrm{CT}, \qquad (3.14)$$

where the contact terms (CT) have to be determined so as to produce a vanishing result for $\mathscr{L}^{(2)}$ when the external magnetic field is switched off. In this limiting case (i.e. $H=0$) we find

$$I_1=\frac{s_1s_2}{s_1+s_2}, \qquad I_2=\frac{2(s_1+s_2)}{(s_1+s_2)^2}$$

$$\mathscr{L}^{(2)}|_{H=0}=-\frac{\mathrm{i}\alpha}{32\pi^3}\int_{\to 0}^\infty ds_1\int_{\to 0}^\infty ds_2\, e^{-\mathrm{i}m^2(s_1+s_2)}\frac{1}{s_1s_2(s_1+s_2)}\left(4m^2-\frac{2\mathrm{i}}{s_1+s_2}\right), \qquad (3.15)$$

which is an infinite constant and has to be subtracted from (3.14).

In order to produce a finite result for (3.14) we also have to subtract the next term in the expansion of the integrand (3.14) which is quadratic in the field, i.e.,

$$-\frac{\mathrm{i}\alpha}{32\pi^3}(eH)^2\int_{\to 0}^\infty ds_1\int_{\to 0}^\infty ds_2\, e^{-\mathrm{i}m^2(s_1+s_2)}\frac{1}{3(s_1+s_2)}\left[2m^2\left(1-\frac{2s_1}{s_2}-\frac{2s_2}{s_1}\right)-\frac{5\mathrm{i}}{s_1+s_2}\right]. \qquad (3.16)$$

Finally we obtain for the second-order radiative correction to the Lagrangian

$$\mathscr{L}^{(2)}=-\frac{\mathrm{i}\alpha}{32\pi^3}\int_{\to 0}^\infty ds_1\int_{\to 0}^\infty ds_2\, e^{-\mathrm{i}m^2(s_1+s_2)}(F(s_1,s_2)-F_0(s_1,s_2)-F_2(s_1,s_2)) \qquad (3.17)$$

where

$$F(s_1,s_2)=\frac{(eH)^2}{s_1\sin(eHs_1)s_2\sin(eHs_2)}[4m^2\cos(eHs_1)\cos(eHs_2)I_1-\mathrm{i}I_2],$$

$$F_0(s_1,s_2)=\frac{1}{s_1s_2(s_1+s_2)}\left(4m^2-\frac{2\mathrm{i}}{s_1+s_2}\right),$$

$$F_2(s_1,s_2)=(eH)^2\frac{1}{3(s_1+s_2)}\left[2m^2\left(1-\frac{2s_1}{s_2}-\frac{2s_2}{s_1}\right)-\frac{5\mathrm{i}}{s_1+s_2}\right].$$

Although the integral (3.17) is still logarithmically divergent as $s_1\to 0$, $s_2\neq 0$, ($s_2\to 0$, $s_1\neq 0$), we may, anticipating the case of very strong magnetic fields, cut off the proper time integral at some lower limit s_0. Without further dwelling on the renormalization procedure, one can demonstrate that in the course of regularization the logarithmic singularity can be isolated and added to the bare electron mass squared so as to define the renormalized mass $m^2=m_0^2+\delta m^2$, where

$$\delta m^2=\frac{3\alpha m^2}{2\pi}\left[\ln\left(\frac{1}{\mathrm{i}\gamma m^2 s_0}\right)+\frac{5}{6}\right].$$

The additive constant $\frac{5}{6}$ follows from Schwinger's work (e.g., Schwinger 1951, appendix B), and $\ln \gamma = C$ is the Eulerian constant.

When H is large, the dominant contribution to the integral comes from the last term in parentheses in equation (3.17). It is mainly this term we want to limit ourselves to in the next section.

4. Strong-field limit of $\mathscr{L}^{(2)}$

Before we investigate the limiting case $eH/m^2 \gg 1$ for $\mathscr{L}^{(2)}$ let us briefly recall the situation for the one-loop effective Lagrangian without radiative correction, $\mathscr{L}^{(1)}$. Here we found the exact result (Dittrich 1976)

$$\mathscr{L}^{(1)} = -\frac{2}{64\pi^2}\Big\{[2m^4 - 4m^2(eH) + \tfrac{4}{3}(eH)^2]\Big[\ln\Big(\frac{m^2}{eH}\Big) + 1\Big] + 4m^2(eH) - 3m^4 + 2(4eH)^2\zeta'\Big(-1;\frac{m^2}{2eH}\Big)\Big\}. \tag{4.1}$$

Keeping terms proportional to $(eH)^2$ only, we obtain

$$\mathscr{L}^{(1)} \to -\frac{2}{64\pi^2}\Big\{\frac{4}{3}(eH)^2\Big[\ln\Big(\frac{m^2}{eH}\Big) + 1\Big] + 32(eH)^2\zeta'\Big(-1;\frac{m^2}{2eH}\Big)\Big\} \tag{4.2}$$

$$\to -\frac{2}{16\pi^2}(eH)^2\Big[\frac{1}{3}\ln\Big(\frac{m^2}{eH}\Big) - \frac{11}{3}\Big]. \tag{4.3}$$

With the expression for the free Lagrangian $\mathscr{L}^{(0)} = -\frac{1}{2}H^2$ one finds the ratio (Weisskopf 1936)

$$\frac{\mathscr{L}^{(1)}}{\mathscr{L}^{(0)}} \cong \frac{\alpha}{\pi}\Big[-\frac{1}{3}\ln\Big(\frac{eH}{m^2}\Big) - \frac{11}{3}\Big]. \tag{4.4}$$

Before looking at the ratio $\mathscr{L}^{(2)}/\mathscr{L}^{(0)}$ we introduce new variables:

$$s(1-u) = eHs_1, \qquad su = eHs_2.$$

Being interested in the dominant F_2 term of equation (3.17) we find

$$\mathscr{L}_2^{(2)} \to \frac{i\alpha}{32\pi^3}\int_{\to 0}^{\infty} s\,ds \int_{\to 0}^{1} du\frac{eH}{3s}\exp\Big(-i\frac{m^2}{eH}s\Big)\Big[2m^2\Big(1 - \frac{2(1-u)}{u} - \frac{2u}{1-u}\Big) - \frac{5i}{s}(eH)\Big].$$

Evidently the $(eH)^2$ dependence is given by the last term which will also provide the logarithm since in this case

$$\mathscr{L}_2^{(2)} \to \frac{\alpha}{32\pi^3}\frac{5}{3}(eH)^2\int_{\to 0}^{\infty}\frac{ds}{s}\int_0^1 du\, e^{-is/b} \underset{s\to -is}{=} \frac{\alpha}{32\pi^3}\frac{5}{3}(eH)^2\int_{\to 0}^{\infty}\frac{ds}{s}e^{-s/b}, \qquad b = \frac{eH}{m^2}.$$

For strong fields only those contributions are important where $1 \ll s \ll b$; here $e^{-s/b} \approx 1$ and we can cut off the integral between $s \approx 1$ and $s \approx b$. Under these conditions we end up with

$$\mathscr{L}^{(2)} \to \frac{\alpha}{32\pi^3}\frac{5}{3}(eH)^2\ln\Big(\frac{eH}{m^2}\Big) = \frac{\alpha^2H^2}{8\pi^2}\frac{5}{3}\ln\Big(\frac{eH}{m^2}\Big). \tag{4.5}$$

Then to within a logarithmic accuracy we find

$$\frac{\mathscr{L}^{(2)}}{\mathscr{L}^{(0)}} \cong \left(\frac{\alpha}{\pi}\right)^2 \left[-\frac{1}{4}\frac{5}{3}\ln\left(\frac{eH}{m^2}\right)\right], \qquad \frac{eH}{m^2} \gg 1. \tag{4.6}$$

The other interesting case, in which only a pure constant electric field is applied, yields the effective Lagrangian $\mathscr{L}^{(0,1,2)}(E) = \mathscr{L}^{(0,1,2)}(H \to \mathrm{i}^{-1}E)$. Hence for strong electric fields:

$$\mathscr{L}(E) = \frac{1}{2}E^2 \left\{1 - \frac{\alpha}{\pi}\frac{1}{3}\left[\ln\left(\frac{eE}{m^2}\right) - \mathrm{i}\frac{\pi}{2} + 11\right] - \left(\frac{\alpha}{\pi}\right)^2 \frac{5}{12}\left[\ln\left(\frac{eE}{m^2}\right) - \mathrm{i}\frac{\pi}{2}\right]\right\}. \tag{4.7}$$

At this stage it is worth noting that the radiative correction to the Coulomb potential shows a similar logarithmic behaviour. However, the modified Coulomb potential is usually arrived at with the aid of the dressed photon propagator which involves the polarization function $\Pi^{(2)}(k^2)$. For large k^2 or for small distances one finds (Gell-Mann and Low 1954) for the electric field

$$E = \frac{e}{4\pi r^2}\left\{1 + \frac{2\alpha}{3\pi}\left[\ln\left(\frac{1}{mr}\right) - C - \frac{5}{6}\right]\right\}, \tag{4.8}$$

in which $C = 0{\cdot}57721\ldots$ is the familiar Eulerian constant. It is tempting, though highly speculative, to employ the constant-field result (4.7) to derive the correction to the Coulomb potential using

$$D(E) = \frac{\partial \mathscr{L}}{\partial E} = E\left[1 - \frac{\alpha}{\pi}\frac{1}{3}\ln\left(\frac{eE}{m^2}\right) - \left(\frac{\alpha}{\pi}\right)^2 \frac{5}{12}\ln\left(\frac{eE}{m^2}\right) + \ldots\right], \tag{4.9}$$

and setting it equal to $e/4\pi r^2$. In fact, Greenman and Rohrlich (1973) take this liberty and apply the Weisskopf–Schwinger Lagrangian to the Coulomb case from which they infer the existence of a minimum electrostatic electron radius. The result of the present paper (e.g., equation (4.7)), could then be used to demonstrate that there is no qualitative change in their arguments, provided such a maximum field strength does exist. However, it is doubtful whether polarization phenomena derived for constant fields can be used to study polarization phenomena in QED at small distances. As was shown in Dittrich (1976), the effect of vacuum polarization differs substantially between various types of electromagnetic fields. If we then utilize the effective Lagrangian of a given external field to derive a modified potential, i.e. a modified coupling constant, one can easily see that, for example, an external plane wave field is certainly not a suitable candidate for generating a modified coupling constant; in fact, the effective Lagrangian reduces to the free one in that special case. It is thus unlikely that the Coulomb potential, constant field or laser field will play identical roles in polarizing the vacuum.

5. Conclusions

In this paper we have re-examined and extended some results associated with the Weisskopf–Schwinger Lagrangian in QED. After setting up the electron's Green function in an external constant magnetic field, we have computed the next higher-order contribution to the one-loop effective potential by including radiative corrections to the electron propagators. The resulting Lagrangian was then investigated for the case of a very strong $H(E)$ field. To within logarithmic accuracy we have shown that in the intense-field limit the effective Lagrangian reveals some similarity with the radiative correction to the Coulomb potential which is usually studied with the polarization function at high momenta. Beyond this formal similarity we must, however, conclude that the effect of polarizing the vacuum is different for various external fields and hence has different consequences concerning the effective coupling constant. The correct way to connect QED at small distances (high k^2) with the strong-field limit of an effective Lagrangian would mean first of all knowing the exact relativistic electron's Green function to all orders in the external Coulomb field and then carrying out the steps that would lead to a Lagrangian similar to that of equation (4.7). Nobody, to the author's knowledge, has ever achieved this goal.

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